

A Small Note About Lower Bound of Eigenvalues*

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Abstract

This paper gives a way to produce the lower bound of eigenvalues defined in a Hilbert space by the eigenvalues defined in another Hilbert space. The method is based on using the max-min principle for the eigenvalue problems.

Keywords. Eigenvalue problem, max-min principle, Hilbert space

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

Recently, there are more and more research about the lower bound of eigenvalues of some type of partial differential operators (see all the reference in this paper). In this paper, we give a framework to produce the lower bound of the eigenvalues defined in some spaces. We would like to say that the results in this not is obtained in our seminar and the motivation is to give a type of framework for the results in [11].

2 Abstract framework

The assumptions for function spaces are listed below

- (A1) Let \mathbb{X} and \mathbb{Y} be two Hilbert spaces with inner product and norm $(\cdot, \cdot)_{\mathbb{X}}$, $\|\cdot\|_{\mathbb{X}}$ and $(\cdot, \cdot)_{\mathbb{Y}}$, $\|\cdot\|_{\mathbb{Y}}$ respectively. And their exists a continuous and compact embedding operator $\gamma : \mathbb{X} \mapsto \mathbb{Y}$.

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(A2) Bilinear form $M(u, v)$ is symmetric, continuous and coercive over the space $\mathbb{X} \times \mathbb{X}$; bilinear form $N(p, q)$ is symmetric, continuous and semi-positive definite over the space $\mathbb{Y} \times \mathbb{Y}$.

Remark 2.1. $M(\cdot, \cdot)$ is an inner product of \mathbb{X} with corresponding norm $\|\cdot\|_M := \sqrt{M(\cdot, \cdot)}$, and $N(\cdot, \cdot)$ is an inner product of $\mathbb{Y} \setminus \ker(N)$ with corresponding norm $\|\cdot\|_N := \sqrt{N(\cdot, \cdot)}$.

In the rest of this paper, for any $x \in \mathbb{X}$, we just use x rather than γx to present the corresponding element in \mathbb{Y} .

Consider the abstract eigenvalue problem: Find $(\lambda, u) \in \mathbb{R} \times \mathbb{X}$, such that $N(u, u) = 1$ and

$$M(u, v) = \lambda N(u, v), \quad \forall v \in \mathbb{X}. \quad (2.1)$$

From the compactness (see, e.g. Section 8 of Babuska Babuska-Osborn-1991), (2.1) has the eigenpairs $\{(\lambda_k, u_k)\}$ ($k = 1, 2, \dots$) with $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $N(u_i, u_j) = \delta_{ij}$ (δ_{ij} : Kronecker's delta).

Let \mathbb{W} and \mathbb{V} be two subspaces of \mathbb{X} . Then we could define the eigenvalue problems on \mathbb{W} and \mathbb{V} , respectively.

Find $(\lambda^{\mathbb{W}}, u^{\mathbb{W}}) \in \mathbb{R} \times \mathbb{W}$, such that $N(u^{\mathbb{W}}, u^{\mathbb{W}}) = 1$ and

$$M(u^{\mathbb{W}}, v^{\mathbb{W}}) = \lambda^{\mathbb{W}} N(u^{\mathbb{W}}, v^{\mathbb{W}}), \quad \forall v^{\mathbb{W}} \in \mathbb{W}. \quad (2.2)$$

Let $\{(\lambda_k^{\mathbb{W}}, u_k^{\mathbb{W}})\}$ ($k = 1, 2, \dots$) be the eigenpairs of (2.2) with $0 < \lambda_1^{\mathbb{W}} \leq \lambda_2^{\mathbb{W}} \leq \dots$ and $N(u_i^{\mathbb{W}}, u_j^{\mathbb{W}}) = \delta_{ij}$.

Find $(\lambda^{\mathbb{V}}, u^{\mathbb{V}}) \in \mathbb{R} \times \mathbb{V}$, such that $N(u^{\mathbb{V}}, u^{\mathbb{V}}) = 1$ and

$$M(u^{\mathbb{V}}, v^{\mathbb{V}}) = \lambda^{\mathbb{V}} N(u^{\mathbb{V}}, v^{\mathbb{V}}), \quad \forall v^{\mathbb{V}} \in \mathbb{V}. \quad (2.3)$$

Let $\{(\lambda_k^{\mathbb{V}}, u_k^{\mathbb{V}})\}$ ($k = 1, 2, \dots$) be the eigenpairs of (2.3) with $0 < \lambda_1^{\mathbb{V}} \leq \lambda_2^{\mathbb{V}} \leq \dots$ and $N(u_i^{\mathbb{V}}, u_j^{\mathbb{V}}) = \delta_{ij}$.

Define $\ker_{\mathbb{X}}(N) := \{x \in \mathbb{X} \mid N(x, x) = 0\}$, $\ker_{\mathbb{W}}(N) := \{w \in \mathbb{W} \mid N(w, w) = 0\}$ and $\ker_{\mathbb{V}}(N) := \{v \in \mathbb{V} \mid N(v, v) = 0\}$. Denote $R(\cdot)$ by the Rayleigh quotient over \mathbb{X} : for any $x \in \mathbb{X} \setminus \ker_{\mathbb{X}}(N)$,

$$R(x) := \frac{M(x, x)}{N(x, x)}. \quad (2.4)$$

Thus the stationary values and stationary points of $R(\cdot)$ over \mathbb{W} and \mathbb{V} correspond to the eigenpairs of eigenvalue problem (2.2) and (2.3), respectively. And the min-max principle holds for both $\lambda_k^{\mathbb{W}}$ and $\lambda_k^{\mathbb{V}}$:

$$\lambda_k^{\mathbb{W}} = \min_{S_k^{\mathbb{W}} \subset \mathbb{W}} \max_{w \in S_k^{\mathbb{W}}} R(w), \quad \lambda_k^{\mathbb{V}} = \min_{S_k^{\mathbb{V}}} \max_{v \in S_k^{\mathbb{V}}} R(v), \quad (2.5)$$

where $S_k^{\mathbb{W}}$ and $S_k^{\mathbb{V}}$ are any k -dimensional subspaces of $\mathbb{W} \setminus \ker_{\mathbb{W}}(N)$ and $\mathbb{V} \setminus \ker_{\mathbb{V}}(N)$, respectively.

Let $P : \mathbb{X} \mapsto \mathbb{V}$ be the projection operator with respect to $M(\cdot, \cdot)$:

$$M(x - Px, v) = 0, \quad \forall x \in \mathbb{X}, \forall v \in \mathbb{V}. \quad (2.6)$$

Then we have the following theorem which is the main result in this note.

Theorem 2.1. *Suppose there exist a constant number α such that the following inequality holds*

$$\|x - Px\|_N \leq \alpha \|x - Px\|_M, \quad \forall x \in \mathbb{X}. \quad (2.7)$$

Let $\lambda_k^{\mathbb{W}}$ and $\lambda_k^{\mathbb{V}}$ be the ones defined in (2.2) and (2.3). Then, we have

$$\frac{\lambda_k^{\mathbb{V}}}{1 + \alpha^2 \lambda_k^{\mathbb{V}}} \leq \lambda_k^{\mathbb{W}} \quad (k = 1, 2, \dots). \quad (2.8)$$

Proof. From the argument of compactness mentioned above, both of the min-max and the max-min principle also hold for λ_k :

$$\lambda_k = \min_{S_k} \max_{u \in S_k} R(u) = \max_{S, \dim(S) \leq k-1} \min_{u \in S^{\mathbb{X}\perp}} R(u), \quad k = 1, 2, \dots, \quad (2.9)$$

where S_k denotes any k -dimensional subspace of $\mathbb{X} \setminus \ker_{\mathbb{X}}(N)$, and $S^{\mathbb{X}\perp}$ denotes the orthogonal complement of S in \mathbb{X} with respect to $M(\cdot, \cdot)$.

Due to the min-max principle, it is clear that $\lambda_k^{\mathbb{W}} \geq \lambda_k$ as $\mathbb{W} \subset \mathbb{X}$. Choosing a special $k-1$ dimensional subspace $\mathbb{V}_{k-1} := \text{span}\{u_1^{\mathbb{V}}, u_2^{\mathbb{V}}, \dots, u_{k-1}^{\mathbb{V}}\}$, we could give a lower bound for λ_k from the max-min principle in (2.9) by

$$\lambda_k^{\mathbb{W}} \geq \lambda_k \geq \min_{v \in \mathbb{V}_{k-1}^{\mathbb{X}\perp}} R(v). \quad (2.10)$$

Let $\mathbb{V}_{k-1}^{\mathbb{V}\perp}$ denotes the orthogonal complement of \mathbb{V}_{k-1} in \mathbb{V} with respect to $M(\cdot, \cdot)$, i.e., $\mathbb{V} = \mathbb{V}_{k-1} \oplus \mathbb{V}_{k-1}^{\mathbb{V}\perp}$. As a consequence, \mathbb{X} can be decomposed by

$$\mathbb{X} = \mathbb{V} \oplus \mathbb{V}^{\mathbb{X}\perp} = \mathbb{V}_{k-1} \oplus \mathbb{V}_{k-1}^{\mathbb{V}\perp} \oplus \mathbb{V}^{\mathbb{X}\perp}. \quad (2.11)$$

Then we have $\mathbb{V}_{k-1}^{\mathbb{X}\perp} = \mathbb{V}_{k-1}^{\mathbb{V}\perp} \oplus \mathbb{V}^{\mathbb{X}\perp}$.

Notice that $\mathbb{V}_{k-1}^{\mathbb{X}\perp} \subset \mathbb{X}$. For any $v \in \mathbb{V}_{k-1}^{\mathbb{X}\perp}$, we have

$$v = Pv + (I - P)v, \quad \text{where } Pv \in \mathbb{V}_{k-1}^{\mathbb{V}\perp}, \quad (I - P)v \in \mathbb{V}^{\mathbb{X}\perp}. \quad (2.12)$$

Then $\|Pv\|_N^2 \leq \frac{\|Pv\|_M^2}{\lambda_k^{\mathbb{V}}}$ is held by

$$\lambda_k^{\mathbb{V}} = \min_{v \in \mathbb{V}_{k-1}^{\mathbb{V}\perp}} R(v) = \min_{v \in \mathbb{V}_{k-1}^{\mathbb{V}\perp}} \frac{\|v\|_M^2}{\|v\|_N^2} \leq \frac{\|Pv\|_M^2}{\|Pv\|_N^2}. \quad (2.13)$$

Therefore, we have for any $v \in \mathbb{V}_{k-1}^{\mathbb{X}\perp}$,

$$\begin{aligned}
R(v) &= \frac{\|v\|_M^2}{\|v\|_N^2} = \frac{\|v\|_M^2}{\|Pv + (I - P)v\|_N^2} \geq \frac{\|v\|_M^2}{\|Pv\|_N^2 + \|v - Pv\|_N^2} \\
&\geq \frac{\|v\|_M^2}{\frac{1}{\lambda_k^\mathbb{V}}\|Pv\|_M^2 + \alpha^2\|v - Pv\|_M^2} \geq \frac{\|v\|_M^2}{(\frac{1}{\lambda_k^\mathbb{V}} + \alpha^2)(\|Pv\|_M^2 + \|v - Pv\|_M^2)} \quad (2.14) \\
&= \frac{\lambda_k^\mathbb{V}\|v\|_M^2}{(1 + \alpha^2\lambda_k^\mathbb{V})(\|Pv\|_M^2 + \|v - Pv\|_M^2)} = \frac{\lambda_k^\mathbb{V}}{1 + \alpha^2\lambda_k^\mathbb{V}}.
\end{aligned}$$

The conclusion in (2.8) is immediately obtained using (2.10) and (2.14). \square

3 Some Applications

Based on Theorem 2.1, it is easy to give the lower-bound results for the eigenvalues which are computed by both the conforming and nonconforming finite element methods if the constant α in (2.7).

Here, we suppose Ω be a domain in \mathbb{R}^d and let V_h^{NC} denote some type of nonconforming finite element space such that $V_h^{\text{NC}} \not\subset V := H_0^1(\Omega)$. If we take $\mathbb{X} := H_0^1(\Omega) + V_h^{\text{NC}}$, $\mathbb{V} = V_h^{\text{NC}}$, $\mathbb{W} = V$,

$$M(u, v) = \int_{\Omega} \nabla u \nabla v d\Omega \quad \text{and} \quad N(u, v) = \int_{\Omega} u v d\Omega.$$

the inequality (2.8) is the result obtained in [5, 11]. For this setting, we can choose CR [3] and ECR [10] elements to build the nonconforming finite element space V_h^{NC} . We would like to say the similar derivatives can be extended to the Biharmonic, Stokes and Steklov eigenvalue problems [4, 7, 8, 9, 11].

When we choose \mathbb{V} as some type of conforming finite element method such that $\mathbb{V} \subset \mathbb{W}$, we can also obtain the lower-bound result (2.8) if we can have the upper bound of the constant α in (2.7).

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